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Wavefunctions and minimum uncertainty states of the harmonic oscillator with an exponentially decaying mass

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Abstract. Exact solutions to the harmonic oscillator with an exponentially decaying mass are explicitly represented in terms of Bessel functions. The dynamical invariant quantity of the system has the form of a rosette-shaped orbit in phase space. From this, it is confirmed that this system is bounded. By using the invariant operator represented in terms of lowering and raising operators, we have obtained wavefunctions and the propagator. Finally, minimum uncertainty states and conditions are evaluated by using other operators which are obtained from the above ones.

1. Introduction

Although there has been much concern about time-dependent quantum systems in recent years, only a few time-dependent Schrödinger equations can be solved exactly. One of these solvable problems is the quantum system with a quadratic Hamiltonian, such as the Caldirola–Kanai Hamiltonian [1, 2]. Dekker [3] investigated the relations between various treatments of the classical linearly damped harmonic oscillator and its quantization. Hagedorn *et al* [4] studied the quantum mechanical evolution generated by the quadratic Hamiltonian with time-dependent coefficients. In particular, considerable effort has been devoted to finding solutions for systems with a time-dependent mass. Papadopoulos [5] obtained the path integral representation of the propagator of a particle with a time-dependent mass. Using a time-dependent canonical transformation, Colegrave and Abdalla [6] used the harmonic oscillator with constant frequency and with variable mass to calculate the electric and magnetic field intensities in a Fabry–Pérot cavity. They also solved the harmonic oscillator with exponentially decaying mass [7] and with strongly pulsating mass [8].

In previous papers, we obtained wavefunctions, energy expectation values, uncertainty relations, transition amplitudes and coherent states for time-dependent quantum systems [9–12]. Moreover, we developed the exact quantum theory of a time-dependent bound quadratic Hamiltonian system [13].

In this paper, we consider the harmonic oscillator with an exponentially decaying mass and a constant spring-constant. The previous results [7] were obtained by using a time-dependent canonical transformation. However, we evaluate the solutions for this system

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by means of changing a time scale. Due to this change, the solution of the classical equation of motion can be explicitly represented in terms of Bessel functions, and we deal with the system by using the invariant operator method. Lewis [14, 15] developed the theory of invariants for treating those quantum systems, which has shed light on their solutions. In addition to their intrinsic mathematical interest, the invariants have received attention because of their use in discussing physical problems [16–18], and their possibility in applications of classical and quantum physics [19].

We shall begin by deriving an invariant operator for the harmonic oscillator with an exponentially decaying mass. In section 2, we obtain an invariant operator and define lowering and raising operators. By using these operators, the eigenstates of the invariant operator and the exact wavefunctions satisfying Schrödinger equation are calculated in section 3. In section 4, the propagator for the system is given by the expansion of the wavefunctions. In section 5, we evaluate the uncertainty relations, the minimum uncertainty states and the minimum uncertainty conditions. Finally, in section 6 we give results and a discussion for the system with an exponentially decaying mass.

2. Invariant quantity

We consider the harmonic oscillator with exponentially decaying mass $M(t) = M_0 e^{-\alpha t}$ in which the parameter (α) is a positive real constant. Since the physical momentum (p_k) is $M_0 \dot{q} e^{-\alpha t}$, the classical equation of motion $(\mathrm{d}p_k/\mathrm{d}t = -kq)$ becomes

$$\ddot{q}(t) - \alpha \dot{q}(t) + \omega_0^2 e^{\alpha t} q(t) = 0 \qquad (2.1)$$

where $\omega_0^2 = k/M_0$. Equation (2.1) is given by the classical Hamiltonian

$$H(p, q, t) = \frac{p^2}{2M_0} e^{\alpha t} + \frac{1}{2} M_0 \omega_0^2 q^2 \qquad (2.2)$$

where p and q are the canonical momentum and coordinate, respectively. To solve the classical equation of motion, we transform the time scale into $s = e^{\alpha t}$. Then the equation of motion (equation (2.1)) becomes

$$\frac{\mathrm{d}^2 q(s)}{\mathrm{d}s^2} + \frac{\omega_0^2}{\alpha^2 s} q(s) = 0. \qquad (2.3)$$

The general solution to equation (2.3) is given by

$$q(s) = s^{1/2} \left[C_1 J_1 \left(\frac{2\omega_0}{\alpha} s^{1/2} \right) + C_2 Y_1 \left(\frac{2\omega_0}{\alpha} s^{1/2} \right) \right] \qquad (2.4)$$

where C_1 and C_2 are constants. Therefore, the solution to equation (2.1) can be written as

$$q(t) = e^{\alpha t/2} [C_3 J_1(z) + C_4 Y_1(z)] \qquad (2.5)$$

where $z = \frac{2\omega_0}{\alpha} e^{\alpha t/2}$ and J_1 and Y_1 are the first- and second-kind Bessel functions of the order of 1, respectively, and C_3 and C_4 are integration constants. We can also represent the solution (equation (2.5)) as

$$q(t) = r(t)[C_5 e^{i\theta(t)} + C_6 e^{-i\theta(t)}] \qquad (2.6)$$

where

$$r(t) = e^{\alpha t/2} [J_1^2(z) + Y_1^2(z)]^{1/2} \qquad (2.7)$$

$$\theta(t) = \arctan[Y_1(z)/J_1(z)]. \qquad (2.8)$$

From equations (2.1) and (2.5) we can easily show that equations (2.7) and (2.8) are solutions of the following differential equations:

$$\ddot{r}(t) - \alpha\dot{r}(t) + [\omega_0^2 \mathrm{e}^{\alpha t} - \dot{\theta}^2(t)]r(t) = 0 \tag{2.9}$$

$$\dot{\theta}(t) = 0. \tag{2.10}$$

Multiplying equation (2.10) by $M_0 r(t)\mathrm{e}^{-\alpha t}$, we can obtain the relations

$$\begin{aligned} 0 &= M_0 r^2(t)\ddot{\theta}(t)\mathrm{e}^{-\alpha t} + 2M_0 r(t)\dot{r}(t)\dot{\theta}(t)\mathrm{e}^{-\alpha t} - \alpha M_0 r^2(t)\dot{\theta}(t)\mathrm{e}^{-\alpha t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}[M_0 r^2(t)\dot{\theta}(t)\mathrm{e}^{-\alpha t}] \\ &\equiv \frac{\mathrm{d}\Omega}{\mathrm{d}t}. \end{aligned}$$

In addition, we can readily confirm that the invariant, Ω, is not zero from the Wronskian of two independent solutions (equation (2.5)),

$$\begin{aligned} \Omega &= M_0 r^2(t)\dot{\theta}(t)\mathrm{e}^{-\alpha t} \\ &= M_0\omega_0 \mathrm{e}^{\alpha t/2}[J_1(z)Y_0(z) - J_0(z)Y_1(z)]. \end{aligned} \tag{2.11}$$

Let us obtain another dynamical invariant quantity, I, satisfying the equation

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \frac{1}{\mathrm{i}\hbar}[I, H] = 0. \tag{2.12}$$

Although there may be many different invariants with many different auxiliary conditions, we take the form of the invariant quantity as

$$I = \tfrac{1}{2}[A(t)p^2 + 2B(t)pq + C(t)q^2] \tag{2.13}$$

where $A(t)$, $B(t)$ and $C(t)$ are real time-dependent functions. From equations (2.2), (2.12) and (2.13) we can obtain the relations of the time-dependent coefficients of I as

$$\dot{A}(t) = -\frac{2B(t)}{M(t)} \tag{2.14}$$

$$\dot{B}(t) = M(t)\omega^2(t)A(t) - \frac{C(t)}{M(t)} \tag{2.15}$$

$$\dot{C}(t) = 2M(t)\omega^2(t)B(t). \tag{2.16}$$

The above equations (equations (2.14)–(2.16)) give the time-dependent coefficients as

$$A(t) = \mathrm{e}^{\alpha t}[J_1^2(z) + Y_1^2(z)] \tag{2.17}$$

$$B(t) = -M_0\omega_0 \mathrm{e}^{\alpha t/2}[J_0(z)J_1(z) + Y_0(z)Y_1(z)] \tag{2.18}$$

$$C(t) = \frac{1}{J_1^2(z) + Y_1^2(z)}\{M_0^2\omega_0^2[J_0(z)J_1(z) + Y_0(z)Y_1(z)]^2 + \mathrm{e}^{-\alpha t}\Omega^2\}. \tag{2.19}$$

From equations (2.13) and (2.17)–(2.19) the dynamical invariant quantity, I, is given by

$$\begin{aligned} I(t) = \frac{1}{2[J_1^2(z) + Y_1^2(z)]}\{&\mathrm{e}^{-\alpha t}\Omega^2 q^2 + \{\mathrm{e}^{\alpha t/2}[J_1^2(z) + Y_1^2(z)]p \\ &- M_0\omega_0[J_0(z)J_1(z) + Y_0(z)Y_1(z)]q\}^2\}. \end{aligned} \tag{2.20}$$

The invariant I can be diagonalized by the transformation matrix

$$T = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{pmatrix} \tag{2.21}$$

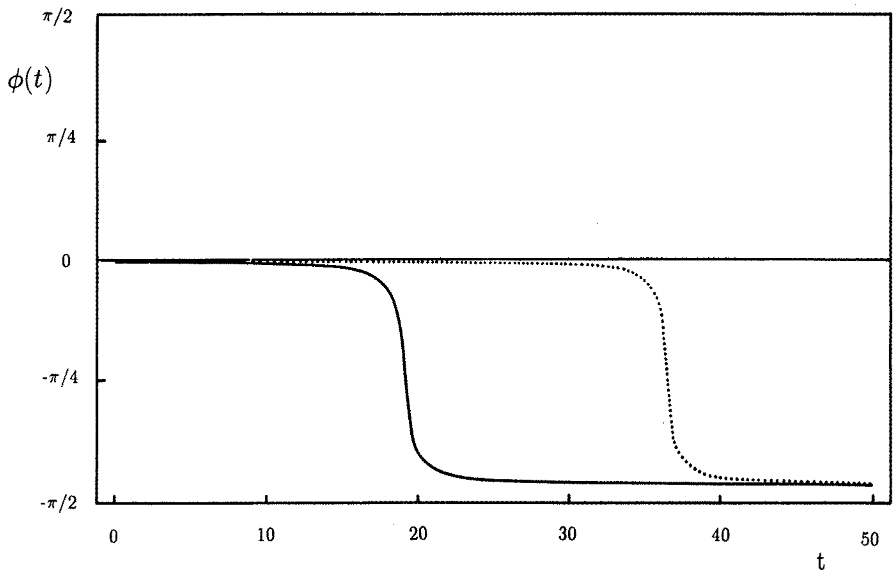

Figure 1. Time variations of the angle $\phi(t)$ between the q-axis and major axis of a rosette-shaped orbit in phase space.

where

$$\phi(t) = \arctan\left\{ -\frac{e^{\alpha t/2}[J_1^2(z)+Y_1^2(z)]}{2M_0\omega_0[J_0(z)J_1(z)+Y_0(z)Y_1(z)]}\delta(t) - \left[1+\frac{e^{\alpha t}}{4M_0^2\omega_0^2}\left(\frac{J_1^2(z)+Y_1^2(z)}{J_0(z)J_1(z)+Y_0(z)Y_1(z)}\right)^2\delta^2(t)\right]^{1/2}\right\} \tag{2.22}$$

$$\delta(t) = 1-\frac{e^{-\alpha t}}{J_1^2(z)+Y_1^2(z)}\{e^{-\alpha t}\Omega^2 - M_0^2\omega_0^2[J_0(z)J_1(z)+Y_0(z)Y_1(z)]^2\}. \tag{2.23}$$

The time-dependent quantity, $\phi(t)$, is the angle between the q-axis and major axis of a rosette-shaped orbit in phase space. The time variations of $\phi(t)$ are illustrated in figure 1. The full and dotted curves correspond to the cases of $\omega_0 = 1.1$ and $\omega_0 = 1.2$, respectively. The angle varies with time from 0 to $-\pi/2$. The form of the dynamical invariant orbit confirms that the harmonic oscillator with an exponentially decaying mass is a bound system.

From equation (2.20) we can define the time-dependent lowering and raising operators, $a(t)$ and $a^\dagger(t)$, as

$$a(t) = \left(\frac{[J_1^2(z)+Y_1^2(z)]}{2\hbar\Omega}\right)^{1/2}\left\{\frac{1}{J_1^2(z)+Y_1^2(z)}[e^{-\alpha t/2}\Omega - iM_0\omega_0(J_0(z)J_1(z) + Y_0(z)Y_1(z))q] + ie^{\alpha t/2}p\right\} \tag{2.24}$$

$$a^\dagger(t) = \left(\frac{[J_1^2(z)+Y_1^2(z)]}{2\hbar\Omega}\right)^{1/2}\left\{\frac{1}{J_1^2(z)+Y_1^2(z)}[e^{-\alpha t/2}\Omega + iM_0\omega_0(J_0(z)J_1(z) + Y_0(z)Y_1(z))q] - ie^{\alpha t/2}p\right\}. \tag{2.25}$$

Then the dynamical invariant operator of the system can be represented in Fock space as

$$I(t) = \hbar\Omega[a(t)^{\dagger}a(t) + \tfrac{1}{2}]. \tag{2.26}$$

3. Wavefunctions

To derive the exact wavefunctions for this system, we calculate first the eigenstate of the invariant operator. Since the lowering operator satisfies relation

$$a(t)U_0(q,t) = 0 \tag{3.1}$$

the ground state is given by

$$\begin{aligned} U_0(q,t) = {} & \left(\frac{M_0\omega_0 \mathrm{e}^{-\alpha t/2}[J_1(z)Y_0(z) - J_0(z)Y_1(z)]}{\hbar\pi[J_1^2(z) + Y_1^2(z)]}\right)^{1/4} \\ & \times \exp\left\{ - \frac{M_0\omega_0\mathrm{e}^{-\alpha t/2}}{2\hbar[J_1^2(z)+Y_1^2(z)]}[(J_1(z)Y_0(z) - J_0(z)Y_1(z)) \right. \\ & \left. -\mathrm{i}(J_0(z)J_1(z) + Y_0(z)Y_1(z))]q^2\right\}. \end{aligned} \tag{3.2}$$

Using equation (3.2), the nth excited states become

$$\begin{aligned} U_n(q,t) = {} & \frac{1}{\sqrt{n!}}[a^{\dagger}(t)]^n U_0(q,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{M(t)\dot{\theta}(t)}{\hbar\pi}\right)^{1/4} \\ & \times \exp\left\{-\frac{M(t)}{2\hbar}\left[\dot{\theta}(t) - \mathrm{i}\frac{\dot{r}(t)}{r(t)}\right]q^2\right\} H_n\left[\sqrt{\frac{M(t)\dot{\theta}(t)}{\hbar}}q\right]. \end{aligned} \tag{3.3}$$

The difference between the eigenstates of an explicitly time-dependent invariant operator and the corresponding solutions to the Schrödinger equation lies only in the time-dependent phase factor [15]. Thus, we can take wavefunctions in the form of

$$\Psi_n(q,t) = U_n(q,t)\mathrm{e}^{\mathrm{i}\vartheta(t)} \tag{3.4}$$

where $\vartheta(t)$ is a real time-dependent function. In addition, equation (3.4) must satisfy the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial\Psi_n(q,t)}{\partial t} = H(p,q,t)\Psi_n(q,t). \tag{3.5}$$

From equations (3.3)–(3.5), the time-dependent phase factor is given by

$$\vartheta(t) = -\left(n+\frac{1}{2}\right)\left\{\arctan[Y_1(z)/J_1(z)] - \arctan\left[Y_1\left(\frac{2\omega_0}{\alpha}\right)\Big/ J_1\left(\frac{2\omega_0}{\alpha}\right)\right]\right\}. \tag{3.6}$$

Consequently, the exact wavefunction is obtained as

$$\begin{aligned} \Psi_n(q,t) = {} & \frac{1}{\sqrt{2^n n!}}\left(\frac{M_0\omega_0 \mathrm{e}^{-\alpha t/2}[J_1(z)Y_0(z) - J_0(z)Y_1(z)]}{\hbar\pi[J_1^2(z) + Y_1^2(z)]}\right)^{1/4} \\ & \times \exp\left\{ - \frac{M_0\omega_0\mathrm{e}^{-\alpha t/2}}{2\hbar[J_1^2(z)+Y_1^2(z)]}\{[J_1(z)Y_0(z) - J_0(z)Y_1(z)] \right. \\ & \left. -\mathrm{i}[J_0(z)J_1(z) + Y_0(z)Y_1(z)]\}q^2\right\} \\ & \times \exp\left\{ - \mathrm{i}\left(n+\frac{1}{2}\right)\left[\arctan(Y_1(z)/J_1(z)) \right.\right. \end{aligned}$$

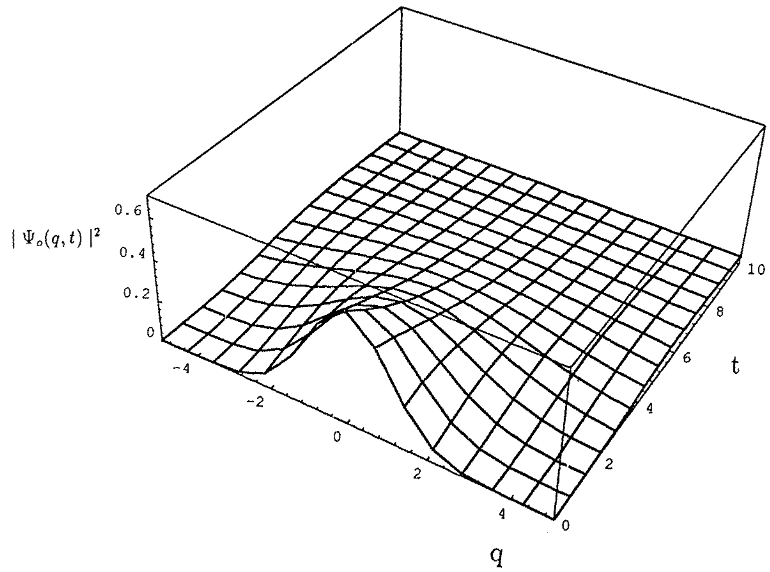

Figure 2. $|\Psi_0(q,t)|^2$ as a function of coordinate (q) and time (t).

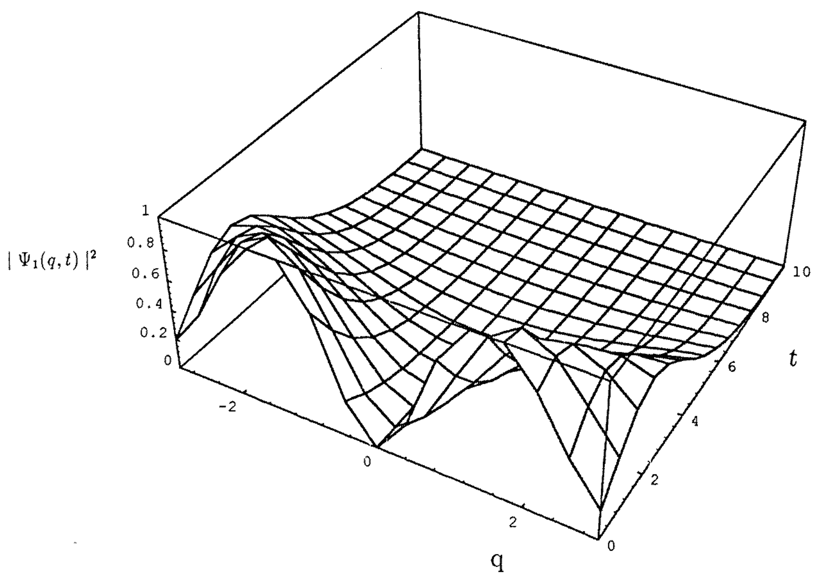

Figure 3. $|\Psi_1(q,t)|^2$ as a function of coordinate (q) and time (t).

$$
-\arctan\left(Y_1\left(\frac{2\omega_0}{\alpha}\right)\Big/J_1\left(\frac{2\omega_0}{\alpha}\right)\right)\Bigg]\Bigg\}
\times H_n\left[\left(\frac{M_0\omega_0 \mathrm{e}^{-\alpha t/2}[J_1(z)Y_0(z)-J_0(z)Y_1(z)]}{\hbar[J_1^2(z)+Y_1^2(z)]}\right)^{1/2} q\right]. \tag{3.7}
$$

In figures 2 and 3, we illustrate the variations of $|\Psi_0(q,t)|^2$ and $|\Psi_1(q,t)|^2$, respectively.

4. Propagator

In the case of a bound system, the propagator is expressed in terms of the time-dependent wavefunctions, $\Psi_n(q,t)$, as [20]

$$K(q,t;q',t') = \sum_{n=0}^{\infty} \Psi_n(q,t)\Psi_n^*(q',t') \tag{4.1}$$

where $\Psi_n(q,t)$ is a wavefunction at time t and coordinate q, and $\Psi_n(q',t')$ is that at time t' and coordinate q'. To find the explicit form of the propagator, we make use of Mehler's formula [21]

$$\sum_{n=0}^{\infty} \frac{Z^n}{n!2^n} H_n(X)H_n(Y) = \frac{1}{\sqrt{1-Z^2}} \exp\left\{\frac{2XYZ-(X^2+Y^2)}{1-Z^2}\right\} \exp(X^2+Y^2). \tag{4.2}$$

Then the propagator of this system is given by

$$K(q,t;q',t') = F(t,t') \exp\{D(t,t')q^2 + E(t,t')q'^2 + G(t,t')qq'\} \tag{4.3}$$

where

$$F(t,t') = \left(\frac{M_0\omega_0}{2\mathrm{i}\hbar\pi}\right)^{1/2} \times \left(\frac{\mathrm{e}^{-\alpha t/2}\mathrm{e}^{-\alpha t'/2}[J_1(z)Y_0(z)-J_0(z)Y_1(z)][J_1(z')Y_0(z')-J_0(z')Y_1(z')]}{[Y_1(z)J_1(z')-J_1(z)Y_1(z')]^2}\right)^{1/4} \tag{4.4}$$

$$D(t,t') = \frac{\mathrm{i}M_0\omega_0\mathrm{e}^{-\alpha t/2}}{2\hbar[J_1^2(z)+Y_1^2(z)]}\left\{[J_0(z)J_1(z)+Y_0(z)Y_1(z)] + \frac{[J_1(z)Y_0(z)-J_0(z)Y_1(z)][J_1(z)J_1(z')+Y_1(z)Y_1(z')]}{Y_1(z)J_1(z')-J_1(z)Y_1(z')}\right\} \tag{4.5}$$

$$E(t,t') = \frac{\mathrm{i}M_0\omega_0\mathrm{e}^{-\alpha t'/2}}{2\hbar[J_1^2(z')+Y_1^2(z')]}\left\{-[J_0(z')J_1(z')+Y_0(z')Y_1(z')] + \frac{[J_1(z')Y_0(z')-J_0(z')Y_1(z')][J_1(z)J_1(z')+Y_1(z)Y_1(z')]}{Y_1(z)J_1(z')-J_1(z)Y_1(z')}\right\} \tag{4.6}$$

$$G(t,t') = \frac{\mathrm{i}M_0\omega_0}{\hbar}\left(\frac{\mathrm{e}^{-\alpha(t+t')/2}[J_1(z)Y_0(z)-J_0(z)Y_1(z)][J_1(z')Y_0(z')-J_0(z')Y_1(z')]}{[Y_1(z)J_1(z')-J_1(z)Y_1(z')]^2}\right)^{1/2}. \tag{4.7}$$

5. Minimum uncertainty state

In this section, the exact uncertainty relations are evaluated at various states. We can define the uncertainty of p and q as

$$(\Delta q\Delta p)_{m,n} \equiv \{[(\langle m|q^2|n\rangle - \langle m|q|n\rangle^2)^*(\langle m|q^2|n\rangle - \langle m|q|n\rangle^2)]^{1/2} \times[(\langle m|p^2|n\rangle - \langle m|p|n\rangle^2)^*(\langle m|p^2|n\rangle - \langle m|p|n\rangle^2)]^{1/2}\}^{1/2}. \tag{5.1}$$

By using the wavefunctions, we can easily show that the average values of p and q vanish. Therefore, the quantum fluctuations are given by

$$\sigma_q \equiv \langle n|q^2|n\rangle - \langle n|q|n\rangle^2$$

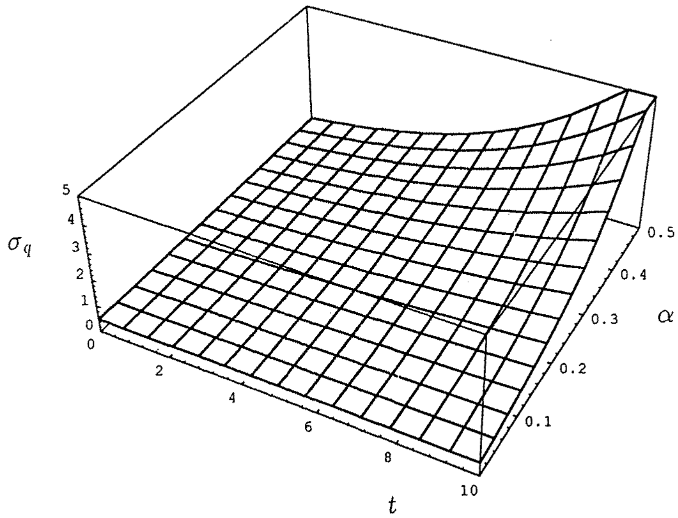

Figure 4. The variations of quantum fluctuation σ_q for the parameter (α) and time (t).

$$= \left(n + \frac{1}{2}\right)\hbar \frac{e^{\alpha t/2}[J_1^2(z) + Y_1^2(z)]}{m_0\omega_0[J_1(z)Y_0(z) - J_0(z)Y_1(z)]} \tag{5.2}$$

and

$$\sigma_p \equiv \langle n|p^2|n\rangle - \langle n|p|n\rangle^2 = \left(n + \frac{1}{2}\right)\hbar \frac{m_0\omega_0 e^{-\alpha t/2}}{[J_1^2(z) + Y_1^2(z)]} \times \{[J_1(z)Y_0(z) - J_0(z)Y_1(z)] - i[J_0(z)J_1(z) + Y_0(z)Y_1(z)]\}. \tag{5.3}$$

In figures 4 and 5, we illustrate the variations of the quantum fluctuations σ_q and σ_p, respectively. Substitution of equations (5.2) and (5.3) into (5.1) gives the uncertainty relations

$$(\Delta q)(\Delta p)_{n,n} = \hbar(n + \tfrac{1}{2})[1 + f(t)]^{1/4} \tag{5.4}$$

where

$$f(t) = \left(\frac{J_0(z)J_1(z) + Y_0(z)Y_1(z)}{J_1(z)Y_0(z) - J_0(z)Y_1(z)}\right)^2. \tag{5.5}$$

In figure 6, we illustrate the variations of function $f(t)$. As α goes to zero, the variable mass is reduced to M_0 and the function $f(t)$ to zero. Therefore, the uncertainty relations are identical with those of a simple harmonic oscillator.

The minimum value in equation (5.4) is larger than $\hbar/2$. Thus, the ground state of this system is not a minimum uncertainty state. To obtain the minimum uncertainty state, we introduce the new lowering and raising operators defined by

$$b(t) = \mu a(t) + \nu a^\dagger(t) \tag{5.6}$$

$$b^\dagger(t) = \mu^* a^\dagger(t) + \nu^* a(t) \tag{5.7}$$

where μ and ν are time-independent c numbers and satisfy the relation $|\mu|^2 - |\nu|^2 = 1$. The canonical transformation (equations (5.6) and (5.7)) which keeps the commutator invariant is a unitary transformation. In a similar way as in section 3, we can obtain the minimum

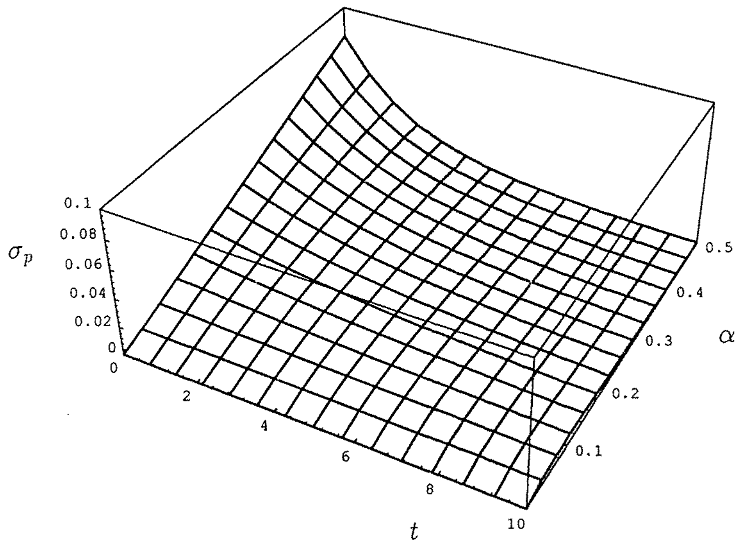

Figure 5. The variations of quantum fluctuation σ_p for the parameter (α) and time (t).

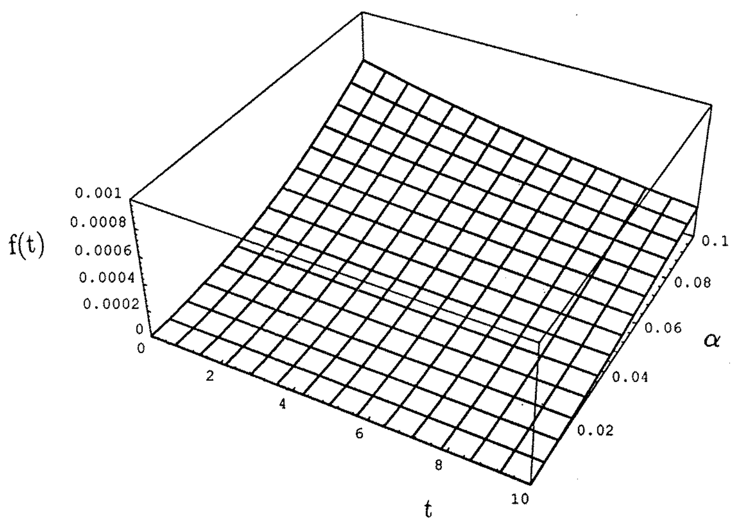

Figure 6. The variations of $f(t)$ in the uncertainty relations as a function of α and time (t).

uncertainty functions

$$\psi_n(q,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{M_0\omega_0 \mathrm{e}^{-\alpha t/2}[J_1(z)Y_0(z) - J_0(z)Y_1(z)]}{\overline{h}\pi|\mu-\nu|^2[J_1^2(z)+Y_1^2(z)]}\right)^{1/4}\left(\frac{\mu^*-\nu^*}{|\mu-\nu|}\right)^n$$
$$\times \exp\left\{-\frac{M_0\omega_0\mathrm{e}^{-\alpha t/2}}{2\overline{h}|\mu-\nu|^2[J_1^2(z)+Y_1^2(z)]}((1-\mu\nu^*+\nu\mu^*)\right.$$
$$\left.\times[J_1(z)Y_0(z) - J_0(z)Y_1(z)] - \mathrm{i}|\mu-\nu|^2[J_0(z)J_1(z)+Y_0(z)Y_1(z)])q^2\right\}$$

$$\times H_n \left[\left(\frac{M_0 \omega_0 \mathrm{e}^{-\alpha t/2} [J_1(z) Y_0(z) - J_0(z) Y_1(z)]}{\hbar |\mu - \nu|^2 [J_1^2(z) + Y_1^2(z)]} \right)^{1/2} q \right]. \tag{5.8}$$

Using equations (5.1) and (5.8), the uncertainty relations for the (n, n) states can be obtained as

$$(\Delta q)(\Delta p)_{n,n} = \hbar (n + \tfrac{1}{2}) \times \left\{ 1 + \left[-|\mu - \nu|^2 \left(\frac{J_0(z) J_1(z) + Y_0(z) Y_1(z)}{J_1(z) Y_0(z) - J_0(z) Y_1(z)} \right) + \mathrm{i}(\mu \nu^* - \nu \mu^*) \right]^2 \right\}^{1/4}. \tag{5.9}$$

To find the minimum uncertainty condition we suggest μ and ν have the form

$$\mu = \frac{k}{\sqrt{k^2 - 1}} \tag{5.10}$$

$$\nu = \frac{1}{\sqrt{k^2 - 1}} \mathrm{e}^{\mathrm{i}\Delta(t)} \tag{5.11}$$

where k is a real and positive constant. From equation (5.9) we can also find the condition of μ and ν for the minimum uncertainty,

$$k = \left(\frac{J_1(z) Y_0(z) - J_0(z) Y_1(z)}{J_0(z) J_1(z) + Y_0(z) Y_1(z)} \right) \beta(t) \pm \left[\left(\frac{J_1(z) Y_0(z) - J_0(z) Y_1(z)}{J_0(z) J_1(z) + Y_0(z) Y_1(z)} \right)^2 \beta^2(t) - 1 \right]^{1/2} \tag{5.12}$$

$$\beta(t) = \frac{J_0(z) J_1(z) + Y_0(z) Y_1(z)}{J_1(z) Y_0(z) - J_0(z) Y_1(z)} \cos \Delta(t) + \sin \Delta(t) \tag{5.13}$$

$$\Delta(t) = \arctan \left\{ \left\{ \left(\frac{J_0(z) J_1(z) + Y_0(z) Y_1(z)}{J_1(z) Y_0(z) - J_0(z) Y_1(z)} \right) \pm \beta(t) \left[\left(\frac{J_0(z) J_1(z) + Y_0(z) Y_1(z)}{J_1(z) Y_0(z) - J_0(z) Y_1(z)} \right)^2 + 1 - \beta^2(t) \right]^{1/2} \right\} [\beta^2(t) - 1]^{-1} \right\} \tag{5.14}$$

and

$$\left(\frac{J_0(z) J_1(z) + Y_0(z) Y_1(z)}{J_1(z) Y_0(z) - J_0(z) Y_1(z)} \right)^2 \leqslant \beta^2(t) \leqslant \left(\frac{J_0(z) J_1(z) + Y_0(z) Y_1(z)}{J_1(z) Y_0(z) - J_0(z) Y_1(z)} \right)^2 + 1. \tag{5.15}$$

We can confirm that the minimum uncertainty is a function of one continuous parameter in the finite region.

6. Results and discussion

The behaviour of nonconservative systems can sometimes be modelled by means of a time-dependent mass. The immediate manifestation of a variable mass occurs in the case of a particle which is disintegrating and losing mass. In this paper, we have obtained quantum mechanical quantities of the harmonic oscillator with an exponentially decaying mass by using the invariant operator method. At this point, we should note that there may be many different invariants with many different auxiliary equations for a given time-dependent Hamiltonian system.

To solve the classical equation of motion for this system, we begin by changing the time scale. Then the solutions can be explicitly represented in terms of the first- and second-kind

Bessel functions. By using these solutions, the quadratic dynamical invariant quantity can be evaluated. The orbit of the invariant quantity has a rosette-shaped form in phase space and confirms that the harmonic oscillator with an exponentially decaying mass is a bound system. The time-dependent angle, $\phi(t)$, between the q-axis and major axis of the orbit is illustrated in figure 1. The full and broken curves correspond to the cases $\omega_0 = 1.1$ and $\omega_0 = 1.2$, respectively, and vary with time from 0 to $-\pi/2$.

By using the lowering and raising operators $a(t)$ and $a^\dagger(t)$, the invariant quantity has been expressed in number states. Although there may be many different operators for a given time-dependent Hamiltonian, we chose the one satisfying equation (2.26). From the application of an invariant operator to the Schrödinger equation, we can readily show that the wavefunctions differ from the eigenstates of invariant operator by a time-dependent phase factor. In figures 2 and 3, we have illustrated the variations of the probability density $|\Psi_0(q,t)|^2$ and $|\Psi_1(q,t)|^2$, respectively. The propagator of the system has been calculated by an expansion of the wavefunctions and by using the Mehler's formula expressed in terms of the nth order Hermite polynomial, $H_n(X)$.

The uncertainty relations at various states are given in equation (5.4). In figures 4–6, the quantum fluctuations and uncertainty relations at various states are illustrated. As α goes to zero, these fluctuations and the function $f(t)$ are reduced to ones of a simple harmonic oscillator with a constant mass. Since the minimum value of the uncertainty for the states (equation (3.7)) is larger than $\hbar/2$, we have evaluated the minimum uncertainty states. To obtain these states, we have carried out unitary transformations such as equations (5.6) and (5.7). In general, new lowering and raising operators $b(t)$ and $b^\dagger(t)$ of another invariant operator $I'(t)$ for different Hamiltonian system can be defined by unitary transformation of $a(t)$ and $a^\dagger(t)$. That is, operators $b(t)$ and $b^\dagger(t)$ in equations (5.6) and (5.7) describe different Hamiltonian system, so $I'(t)$ is different from $I(t)$. This new invariant operator, $I'(t)$, may be constituted by $b(t)$ and $b^\dagger(t)$ in the form of $I'(t) = \hbar\Omega'[b(t)^\dagger b(t) + \frac{1}{2}]$. In a similar way as in section 3, we have evaluated the ground state and the nth excited state of this invariant operator, $I'(t)$, by using $b(t)$ and $b^\dagger(t)$. That is, equation (5.8) gives the eigenstates of this invariant operator, $I'(t)$, which are the minimum uncertainty states with the minimum uncertainty condition of equation (5.15). Under this condition, therefore, the uncertainty relations for these states $\psi_n(q,t)$ have been obtained by $(\Delta q\Delta p)_{n,n} = \hbar(n+\frac{1}{2})$.

In future work, we shall evaluate more directly the exact solutions (invariant quantity, exact wavefunctions, propagator and minimum uncertainty states) for the harmonic oscillator with a pulsating mass.

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